March 25: Nagata Rings, part 2

Integral closure in characteristic zero

Theorem F. Suppose that R has characteristic zero and satisfies condition N_1. Then R satisfies N_2. In particular, if R is integrally closed, and has characteristic zero, then R satisfies N_2.

Proof. Let L be a finite extension of K and S the integral closure of R in L.

If we show that S is a finite R'-module, then since R satisfies N_1, S is a finite R-module.

Thus, it suffices to prove the second statement.

In fact: At this point we do not need to assume R has characteristic zero, only that the extension is separable.
We now assume R is integrally closed. Since L is a separable extension of K, we may enlarge L to a Galois extension L' of K.

If the integral closure of R in L' is finite over R, then S is finite over R.

Thus, WLOG, we assume L is Galois over K.

Integral closure in characteristic zero, continued

Since L is separable over K, we may write $L = K(\alpha)$, for some $\alpha \in L$.

In fact, we may take $a \in S$, such that $L = K(a)$, by clearing denominators in an equation of algebraic dependence for α over K.

Recall: Since R is integrally closed, the minimal polynomial $f(x)$ for a over K has coefficients in R. Let $a = a_1, a_2, \dots a_n$ be the roots of $f(x)$.

Thus, n is the degree of $f(x)$ and every element in L can be written (uniquley) in the form:

$$k_0 1 + k_1 a + \cdots + k_{n-1} a^{n-1},$$

for $k_j \in K$.

Integral closure in characteristic zero, continued

Let $\sigma_1, \dots, \sigma_n$ denote the elements of the Galois group of L over K. Set $d := \prod_{i<j}(\sigma_i(a) - \sigma_j(a))^2$, the *discriminant* of $f(x)$. The proof is complete if we show $d \cdot S \subseteq R[a]$.

Let $s \in S$ and write

$$s = k_0 1 + k_1 a + \cdots + k_{n-1} a^{n-1}, \qquad (*)$$

with $k_j \in K$. If we show that $d \cdot k_j \in R$, for all j, then $d \cdot s \in R[a]$.

Applying each σ_i to (*), we get an $n \times n$ system of equations of the form

$$\sigma_i(s) = k_0 1 + k_1 \sigma_i(a) + \cdots + k_{n-1} \sigma_i(a)^{n-1}. \qquad (**)$$

This yields a matrix equation $A \cdot \begin{bmatrix} k_0 \\ \vdots \\ k_{n-1} \end{bmatrix} = \begin{bmatrix} \sigma_1(s) \\ \vdots \\ \sigma_n(s) \end{bmatrix}$, where $A = (\sigma_i(a)^j)$.

Integral closure in characteristic zero, continued

Let $\tilde{A}$ denote the adjugate of A. We note: (i) A is a Van der Mond matrix. Thus $\det(A)^2 = d$.

(ii) Each $\sigma_i(s)$ and $\sigma_i(a)^j$ is integral over R, and thus belongs to S.

(iii) Multiplying (**) by $\tilde{A}$ shows the entries of $\tilde{A} \cdot \begin{bmatrix} \sigma_1(s) \\ \vdots \\ \sigma_n(s) \end{bmatrix}$ are integral over R.

(iv) Thus each $\det(A) \cdot k_i$ is integral over R. Therefore $d \cdot k_i$ is integral over R.

(v) On the other hand, for each i, $\sigma_j(dk_i) = dk_i$, for all j. Thus $dk_i \in K$, for all i.

(vi) Since each dk_i is integral over R, each $dk_i \in R$, as required.

Nagata's example of a non-excellent DVR

The example below due to Nagata constructs a one-dimensional local domain S with infinite integral closure and also a one-dimensional DVR R that does not satisfy N_2. Since an excellent local domain must satisfy N_2, R is not excellent.

We start with a field K of characteristic $p > 0$ such that $[K : K^p] = \infty$. For example, one can take K to be $\mathbb{Z}_p(U_1, U_2, \ldots)$, where $\{U_i\}$ are algebraically independent over $\mathbb{Z}_p$.

We set $T := K[[x]]$ and $R := K^p[[x]][K]$, where x is analytically independent over K. We follow the steps below.

Step 1. For $f = \sum_{i=0}^{\infty} \alpha_i x^i \in T$, $f \in R$ if and only if $[K^p(\{\alpha_i\}) : K^p] < \infty$.

Proof. Suppose $f \in R$. Then we can write $f = g_1 k_1 + \cdots + g_r k_r$, for $g_j \in K^p[[x]]$ and $k_j \in K$. If we write $g_j := \sum_{i=0}^{\infty} \beta_{ij} x^i$,

then for all $i \geq 0$, we have $\alpha_i = \beta_{i1} k_1 + \cdots + \beta_{ir} k_r$. It follows that $K^p(\{\alpha_i\}) \subseteq K^p \cdot k_1 + \cdots + K^p \cdot k_r$, and thus $[K^p(\{\alpha_i\}) : K^p] < \infty$.

Nagata's example, continued

Conversely, suppose $[K^p(\{\alpha_i\}) : K^p] < \infty$. Let $k_1, \ldots, k_r$ be a basis for $K^p(\{\alpha_i\})$ over K^p.

Then for each $i \geq 0$, we have an equation $\alpha_i = \beta_{i1}k_1 + \cdots + \beta_{ir}k_r$, with each $\beta_{ij} \in K^p$.

It follows that if we set $g_j := \sum_{i=0}^{\infty} \beta_{ij}x^i$, then $f = g_1k_1 + \cdots + g_rk_r$, and hence $f \in R$.

Step 2. R is a DVR.

Proof. It suffices to show that xR is the set of non-units of R (and hence xR is the unique maximal ideal of R) and $\bigcap_{i=1}^{\infty} x^nR = 0$.

The second statement follows since $\bigcap_{i=1}^{\infty} x^nT = 0$.

Nagata's example, continued

For the first statement, suppose $f = \sum_{i=0}^{\infty} \alpha_i x^i \in R$ is a non-unit. We claim $\alpha_0 = 0$.

Suppose not. Then f is a unit in T, and hence there exists $g = \sum_{i=0}^{\infty} \beta_i x^i$ such that $fg = 1$.

If we solve the resulting system of equations:

$$\begin{aligned} \alpha_0\beta_0 &= 1 \\ \alpha_1\beta_0 + \alpha_0\beta_1 &= 0 \\ &\vdots \end{aligned}$$

for the β_i, we see that $K^p(\{\beta_i\}) \subseteq K^p(\{\alpha_i\})$, and thus $[K^p(\{\beta_i\}) : K^p] < \infty$, since $f \in R$.

Thus, $g \in R$, which is a contradiction, since f is a non-unit in R.

Nagata's example, continued

It follows that $\alpha_0 = 0$. Therefore, we can write $f = x\tilde{f}$.

Since the coefficients of $\tilde{f}$ are the same as the coefficients of f, only shifted by one degree, by Step 1, $\tilde{f} \in R$.

Thus, $f \in xR$. Therefore, the non-units of R are contained in xR.

Since every element of xR is clearly a non-unit in R, it follows that xR is the set of non-units in R. Thus, R is a DVR.

Step 3. T is the xR-adic completion of R.

Proof. Every f in T in the limit (in the xT-adic topology) of a sequence of polynomials $\{f_n\} \subseteq K[x]$. Each $f_n \in R$.

Thus, R is dense in T. Since T is complete in the xT-adic topology, T is the completion of R.

Nagata's example, continued

Step 4. Take $\beta_0, \beta_1, \ldots \in K$ such that $[K^p(\{\beta_i\}) : K^p] < \infty$ and set $a := \sum_{i=0}^{\infty} \beta_i x^i$, so $a \in T \backslash R$. Then $a^p \in R$ and for an indeterminate Y over R , $Y^p - a^p$ is irreducible over R.

Proof. In T, we have $a = \lim_{n\to\infty} a_n$, where

$$a_n := \beta_0 + \cdots + \beta_n x^n.$$

Thus, $\lim_{n\to\infty} a_n^p = a^p$. Since $a_n^p = \beta_0^p + \cdots + \beta_n x^{np}$, it follows that $a^p = \sum_{i=0}^{\infty} \beta_i^p x^{ip} \in R$.

Now, a^p is not a pth power in R, otherwise $a^p = r^p$, for some $r \in R$, and thus $(a - r)^p = 0$, so $a - r = 0$, which gives $a \in R$, a contradiction.

Since R is integrally closed a^p is not a pth power in the quotient field of R, so $Y^p - a^p$ is irreducible over R.

Nagata's example, continued

Step 5. Set $S := R[a]$. Then $S \cong R[Y]/(Y^p - a^p)$ and S is a one-dimensional local domain whose integral closure S' is not a finite S-module.

Proof. From the previous step we know that $Y^p - a^p$ generates a height one prime in the UFD $R[Y]$. Since $Y^p - a^p$ belongs to the kernel of the natural map from $R[Y]$ to $R[a]$, it must generate the kernel. This gives the first statement.

For the second statement, S is integral over R, so it is one-dimensional. Moreover, $h^p \in R$, for all $h \in S$, so S must local.

To see this, suppose $Q_1, Q_2 \subseteq S$ are two maximal ideals. Since R is local, $Q_1 \cap R = Q_2 \cap R$. Take $h \in Q_1 \backslash Q_2$.

Then $h^p \in Q_1 \cap R = Q_2 \cap R$, so $h^p \in Q_2$. Thus, $h \in Q_2$, a contradiction.

Therefore S is a one-dimensional local domain.

Nagata's example, continued

We now claim $\widehat{S} \cong \widehat{R}[Y]/(Y^p - a^p) = T[Y]/(Y^p - a^p)$. Here the completions of R and S are taken with respect to the x-adic topology, which in each case yields the completion with respect to the respective maximal ideals.

Suppose the claim holds. In $T[Y]$, $Y^p - a^p = (Y - a)^p$, which shows that $T[Y]/(Y^p - a^p)$ and hence $\widehat{S}$ is reduced.

Thus, S is not analytically unramified. By what we have previously shown in class, this implies that S' is not a finite S-module.

Nagata's example, continued

For the claim, we tensor the exact sequence

$$0 \to (Y^p - a^p)R[Y] \xrightarrow{i} R[Y] \to S \to 0$$

with $\widehat{R}$ to obtain the exact sequence

$$0 \to (Y^p - a^p)R[Y] \otimes \widehat{R} \xrightarrow{\hat{i}} \widehat{R}[Y] \to \widehat{S} \to 0,$$

where we use the easy-to-check fact that $R[Y] \otimes \widehat{R} = \widehat{R}[Y]$. Since the image of the map $\hat{i}$ is $(Y^p - a^p)\widehat{R}[Y]$, this yields the claim.

Step 6. R does not satisfy N_2. Hence R is a non-excellent DVR.

Proof. Since a finite extension of a ring satisfying N_2 must have a finite integral closure, the first statement follows from the previous step. The second statement follows from the fact that an excellent local domain must be a Nagata domain, and hence must satisfy N_2.

Remark. The example above is a special case of Nagata's example, in that Nagata takes more variables. In other words, he sets

$$T := K[[x_1, \ldots, x_d]] \text{ and } R := K^p[[x_1, \ldots, x_d]][K],$$

where $x_1, \ldots, x_d$ are analytically independent variables over F.

Nagata proves that R is a regular local ring with completion T. When $d = 2$ and $d = 3$, he uses R and T to also construct:

(a) An example of a two-dimensional Noetherian domain A and a non-Noetherian ring B such that $A \subseteq B \subseteq A'$ and

(b) An example of a three-dimensional Noetherian domain C such that C' is not Noetherian.

These examples are relevant because on the one hand, every ring between a one-dimensional Noetherian domain and its quotient field is Noetherian, while on the other hand, the integral closure of any two-dimensional Noetherian domain is Noetherian.